Exact Consumer's Surplus and Deadweight Loss

By JERRY A. HAUSMAN*

Consumer's surplus is a widely used tool in applied welfare economics. Both economic theorists and cost benefit analysis often use consumer's surplus despite its somewhat dubious reputation. The basic idea is to evaluate the value to a consumer or his "willingness to pay" for a change in price of a good from say price p^0 to price p^1. Because price changes affect consumer welfare, an evaluation of this effect is often a key input to public policy decisions. Yet consumer's surplus is probably the most controversial of widely used economic concepts. Both Paul Samuelson and Ian Little conclude that the economics profession would be better off without it.

It is my feeling of the situation that substantial agreement exists on the correct quantities to be measured: the amount the consumer would pay or would need to be paid to be just as well off after the price change as he was before the price change. The quantities correspond to John Hicks' compensating variation measures. An alternative measure which takes *ex post* price change utility as the basis of comparison is Hicks' equivalent variation.[1] The controversy arises in the measurement of these quantities. The usual measurement procedure is to use the area to the left of the Marshallian (market) demand curve between two price levels. Jules Dupuit originated this measure of welfare change, and Alfred Marshall and Hicks derived appropriate conditions for its use. The primary condition for the area to the left of the demand curve to correspond to the compensating variation is to have constant marginal utility of income. Marshall gave this condition, and if it holds, the same quantity will be derived as the area to the left of the compensated (Hicksian) demand curve. This area to the left of the compensated demand curve is exactly what the compensating variation and equivalent variation measure. Thus the constant marginal utility of income is a sufficient condition for Marshallian consumer's surplus to be equal to Hicks' consumer's surplus. In this case Arnold Harberger's plea to use the welfare triangle as one-half times the product of the price change times the quantity change to measure deadweight loss corresponds to the correct theoretical amount of welfare change.

In a recent paper, Robert Willig derives bounds for the percentage difference between the correct measure of either the compensating or equivalent variation and the Marshallian measure derived form the market demand curve. His bounds, which depend on the income elasticity of demand for the single good in the region of price change being considered as well as the proportion of the consumer's income spent on the good, demonstrate that the Marshallian consumer's surplus is often a good approximation to Hicks' consumer's surplus. The fact that the proportion of the consumer's income spent matters as well as the income elasticity was first pointed out by Harold Hotelling. Willig contends that the approximation error will be less than the errors involved in estimating the demand curve. Thus he hopes to remove the need for apology that applied economists often need to give to theorists who remark on the inappropriateness of using Marshallian consumer's surplus to measure welfare change.

However, in this paper I show that for the case primarily considered by Willig of a single price change, which is also the situation in which consumer's surplus is often used in applied work, no approximation is necessary.

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[1]The reason that we still have two, rather than one, of Samuelson's six measures of consumer's surplus arises from an index number problem of the correct basis for the welfare comparison. I will give both measures but plan to concentrate on the compensating variation.

From an estimate of the demand curve, we can derive a measure of the *exact* consumer's surplus, whether it is the compensating variation, equivalent variation, or some measure of utility change. No approximation is involved. While this result has been known for a long time by economic theorists, applied economists have only a limited awareness of its application. Furthermore, for the majority of cases the calculations are simple enough for a hand calculator. It seems preferable to remove completely any approximation argument from so important a matter as consumer's surplus. Also, my exact formulae allow calculation of the precision of our estimated consumer's surplus in terms of a standard error of estimation. Since unknown parameters for the demand curve will usually be estimated by econometric procedures, standard error formulae allow construction of confidence regions for the estimated compensated variation. These confidence regions might well be an important input to policy decisions. In most empirical applications we would like to account for the error in estimating the demand curve rather than including it in the approximation error as Willig implicitly does. Lastly, for some important uses of consumer's surplus, Willig's approximation argument is not useful. For instance, in assessing the welfare loss from taxation of labor income or capital income the proportion of total income can become so large that the Marshallian measure could differ markedly form the Hicks' measure of compensating variation or equivalent variation.[2]

However, a more important shortcoming of the use of the Marshallian measure (and Willig's approximation argument) arises in measuring deadweight loss. Here we are not interested in the complete compensating variation, which is a trapezoid to the left of the appropriate demand curve, but rather the triangle which corresponds to the excess of the compensating variation over the tax revenue collected from an individual. This triangle corresponds to the welfare measure that Harberger has used in his many studies of the effect of taxation on the U.S. economy. Even in cases where Willig's approximations hold for the complete compensating variation, the Marshallian deadweight loss can be a very poor approximation for the theoretically correct Hicksian measure of deadweight loss based on the compensated demand curve. Thus the Marshallian measure of deadweight loss is not accurate for the important measurements often undertaken in applied welfare economics and public finance studies. But, again, given an estimate of the uncompensated demand curve we can derive the exact measure of deadweight loss. As the example in the concluding section of the paper shows, the traditional measurement of the welfare triangle can lead to badly biased estimates of the true deadweight loss even when the conditions for Willig's approximation argument hold true for measurement of consumer's surplus.

The basic idea used in deriving the exact measure of consumer's surplus is to use the *observed* market demand curve to derive the *unobserved* compensated demand curve. It is this latter demand curve which leads to the compensating variation and equivalent variation.[3] In the two-good case using modern duality theory, I begin with the market demand curve and derive the corresponding indirect utility function. These two functions permit exact calculation of the compensating variation, equivalent variation and deadweight loss. In the many-good case when a single price changes, I derive the "quasi" indirect utility function and the "quasi" expenditure function. I denote the appropriate functions as quasi since they do not corre-

[2] For recent uses of consumer's surplus in these situations, see Michael Boskin and Martin Feldstein. Many important applications in public finance have the feature that a large proportion of an individual's income is involved.

[3] Hal Varian derives the compensating variation as the area under the Hicksian compensated demand curve. He then remarks that "unfortunately, since the Hicksian demand curves are unobservable these expressions do not appear to be useful" (p. 210). Herbert Mohring considers the properties of different welfare measures and uses a technique similar to mine to derive the compensating variation for the Cobb-Douglas case. G. W. McKenzie and I. F. Pearce and Y. O. Vartia use somewhat similar approaches but use different methods of analysis.

spond exactly to the individual's indirect utility function and expenditure function. To derive these functions, one would require estimates of the complete system of demand equations. The complete demand system usually cannot be estimated due to lack of data. Instead, I use Hicks' aggregation theorem to demonstrate that the quasi functions which correspond to the assumption of a two-good world would give exactly the same measure of consumer's surplus as the actual functions for a single price change. Thus, the estimates of the uncompensated demand curve are all that is required to produce estimates which correspond to the correct theoretical magnitude.

My approach differs from much recent work in that I begin with the observed market demand curve and then derive the unobserved indirect utility function and expenditure function. The more common approach is to start from a specification of the utility function, for example, Stone-Geary or translog, and then estimate the unknown parameters from the derived market demand functions. The method used here seems preferable on two grounds. First, the only observable data are the market demand data so good econometric practice would indicate finding a function that fits the data well. Thus, different specifications of the demand curve, not the utility function, would be fit with the best-fitting demand equation chosen to base the applied welfare analysis on. Second, specifications such as the translog functions force all the demand curves to have the same functional form which are often difficult to fit econometrically. Since here I consider only partial-equilibrium welfare analysis, I need only estimate a single demand function. Again, alternative specifications of the demand curve allow consideration of the robustness of the results to the chosen specification. The demand curve approach offers considerably more flexibility than does the utility function approach in obtaining good econometric results given the available market data.

In the next section, I derive the indirect utility function and expenditure function for the two-good case. It is shown how the use of these functions leads to correct measure of the compensating variation and equivalent variation. Section II then extends the analysis to the many-good case when only one price changes. There I show that the two-good analysis can be applied with only slight modifications. The functions for the case of a general quadratic demand curve are also derived. Lastly, in Section III, I provide an example of labor supply where the Marshallian approximation is inaccurate for the true compensating variation. I also provide an example of the calculation of deadweight loss to demonstrate that even when the Marshallian measure of the compensating variation is reasonably accurate, the Marshallian measure of deadweight loss can be incorrect by a relatively large amount. Section IV provides a brief conclusion to the paper.

I. The Compensating Variation and Equivalent Variation in the Two-Good Case

The basic tools which I will use in the analysis emerge from the dual approach to consumer behavior. The conventional treatment of consumer behavior considers the maximization of a strictly quasi-concave utility function defined over n goods, $x=(x_1,\ldots,x_n)$, subject to a budget constraint.

$$(1)\quad \max_x u(x) \text{ subject to } \sum_{i=1}^{n} p_i x_i = p\cdot x \leqslant y$$

where p_i are prices and y is (nonlabor) income.[4] The dual approach to the problem is to consider the associated minimization problem which defines the expenditure function

$$(2)\quad e(p,\bar{u}) \equiv \min_x p\cdot x \text{ subject to } u(x) \geqslant \bar{u}$$

The expenditure function was introduced into the literature by Lionel McKenzie; for recent

[4]Local nonsatiation will be assumed throughout the analysis so that the budget constraint will hold as an equality.

analysis and applications see Leo Hurwicz and Hirofumi Uzawa and Peter Diamond and Daniel McFadden. Charles Blackorby and W. Erwin Diewert have recently studied local properties of the expenditure function. The important property of the expenditure function which we will find extremely useful is that the partial derivative with respect to the jth price gives the Hicksian compensated demand curves.[5]

$$\frac{\partial e(p,\bar{u})}{\partial p_j}=h_j(p,\bar{u}) \tag{3}$$

These unobservable Hicksian demand curves should be distinguished from the observable market uncompensated demand curves $x(p, y)$. At an optimum solution to equations (1) and (2) the demands coincide at maximum utility u^*, $h(p,u^*)=x(p,y)$.

The other function we will use which connects the utility function of equation (1) and the expenditure function of equation (2) is the indirect utility function which is the solution to the maximization problem

$$v(p,y)\equiv max\left[u(x): p\cdot x\leqslant y\right] \tag{4}$$

Properties of the indirect utility function are derived in Diewert. An important property of the indirect utility function which we will use is René Roy's identity which yields the observed market demand curves as partial derivatives of $v(p, y)$.

$$x_j(p,y)=-\partial v(p,y)/\partial p_j/\partial v(p,y)/\partial y \tag{5}$$

It is the difference between equation (3) for the compensated demand curve and equation (5) for the uncompensated demand curve that induces the difference between Marshallian consumer's surplus and exact Hicks' consumer's surplus when a price change occurs. Since the indirect utility function of equation (4) is monotonically increasing in income while the expenditure function of equation (2) is monotonically increasing in utility, either function can be inverted to derive the other corresponding function.

Let us now consider a change in the price vector from p^0 to p^1 and formally define the exact measures of consumer's surplus, the compensating variation, and equivalent variation, using the expenditure function.[6] Holding nonlabor income constant at y^0, the compensating variation $CV\ (p^0, p^1, y^0)$ is the minimum quantity required to keep the consumer as well off as he was in the initial state characterized by (p^0, y^0) as he is in the new state (p^1, y^0+CV). In terms of the expenditure function

$$\begin{aligned} CV(p^0,p^1,y^0)&=e(p^1,u^0)-e(p^0,u^0)\\ &=e(p^1,u^0)-y^0 \end{aligned} \tag{6}$$

where $u^0=v(p^0,y^0)$ from the indirect utility function. Equivalently the compensating variation can be defined through the indirect utility function as $v(p^1, y^0+CV)=v(p^0,y^0)$. An alternative measure of welfare change is the equivalent variation, $EV(p^0, p^1, y^0)$, which uses utility after the price change as the basis of comparison:[7]

$$EV(p^0,p^1,y^0)=e(p^1,u^1)-e(p^0,u^1) \tag{7}$$

Using either the compensating variation or equivalent variation, it can be shown that the area under the compensated Hicksian demand curve corresponds to consumer's surplus.

[5]The other useful property of the expenditure function which will be utilized in subsequent analysis is that the second derivatives of the expenditure function yield the elements of the Slutsky matrix $S_{ij}=\partial^2 e(p,\bar{u})/\partial p_i \partial p_j=\partial h_j(p,\bar{u})/\partial p_i$.

[6]Willig and Avinash Dixit and P. A. Weller do a similar derivation.

[7]The compensating variation and equivalent variation always have the same sign because of the monotonicity of $e(p,u)$ in prices so long as the net demands do not change sign. Except for the single price change case, no inequality relationship holds in general.

Let us consider the case when only the first price changes from p_1^0 to p_1^1 with all other prices held constant. Equation (3) gives the compensated demand curve, and integrating it between the two price levels gives

$$(8)\quad CV(p^0, p^1, y^0) = e(p^1, u^0) - e(p^0, u^0)$$

$$= \int_{p_1^0}^{p_1^1} h_1(p, u^0)\,dp_1 = \int_{p_1^0}^{p_1^1} \frac{\partial e(p, u^0)}{\partial p_1}\,dp_1$$

The equivalent variation is derived in an identical manner where u^0 is replaced by u^1.[8]

Let us now compare this measure of welfare change with the traditional measure of Marshallian consumer's surplus as the area under the uncompensated demand curve of equation (5).[9] The integral has the form

$$(9)\quad A(p^0, p^1, y^0) = \int_{p_1^0}^{p_1^1} x_1(p, y^0)\,dp_1$$

$$= -\int_{p_1^0}^{p_1^1} \frac{\partial v(p, y^0)/\partial p_1}{\partial v(p, y^0)/\partial y}\,dp_1$$

This integral in general differs from the integral for the compensating variation in equation (8). To keep the individual on the same indifference curve, y^0 which enters both the numerator and denominator of equation (9) must be constantly adjusted along the path of the price change. Since y^0 is kept constant, this produces the difference between the uncompensated market demand curve with its Marshallian measure of consumer's surplus and the compensated demand curve with its measure of the compensating variation.

It is the supposed constancy or near constancy of the marginal utility of income which has often served as a basis for using Marshallian consumer's surplus as a measure of welfare change. However, equations (6) and (9) in general do not give the same measure. The difference between the compensated Hicksian demand curve which forms the basis for equation (6) and the uncompensated Marshallian demand curve which forms the basis for equation (9) follows from Slutsky's equation

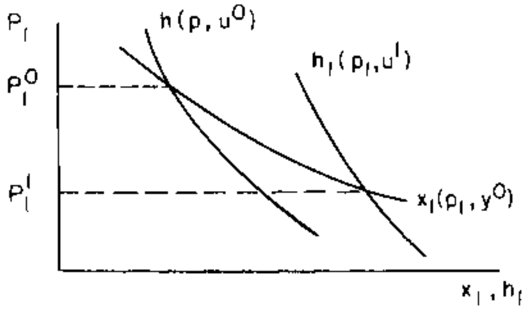

FIGURE 1

$$(10)\quad \frac{\partial h_1(p, u^0)}{\partial p_1} - \frac{\partial x_1(p, y^0)}{\partial p_1} = x_1 \cdot \frac{\partial x_1(p, y^0)}{\partial y}$$

A sufficient condition for equation (10) to equal zero is that both $\partial^2 v(p, y^0)/\partial y \partial p$ and $\partial^2 v(p, y^0)/\partial y^2$ equal zero. These conditions correspond to the case of constant marginal utility of income. For the case of a normal good, the compensated demand curve has steeper slope than the market demand curve so Figure 1 demonstrates the inequalities for a single price change $EV(p^0, p^1, y^0) \leq A(p^0, p^1, y^0) \leq CV(p^0, p^1, y^0)$, an inequality found in Willig. His paper shows that even when the marginal utility of income is not constant that the percentage difference, $(CV - A)/A$, is not large under certain conditions.

Let us now turn to the empirical application of consumer's surplus. It turns out that for many applications no approximation is needed since equation (6) or (7) can be computed exactly. I begin with the simplest case, two goods only with prices $p^0 = (p^0, 1)$. Thus I use the second good as numeraire and consider a price change to p_1^1. Both the price of the first good and income are normalized with respect to the price of the second good,

[8]An alternative but equivalent method of interpreting our procedure is to use equation (3) to write $\partial e/\partial p_j = h_j(p, \bar{u}) = x_j(p, e(p, \bar{u}))$. In principle this implicit equation can always be numerically integrated from p^0 to p^1 to find the exact compensating variation. Vartia gives a computer algorithm for the numerical integration method. My technique to find closed-form solutions uses Roy's identity to derive a differential equation which can be explicitly solved in many cases.

[9]Varian (pp. 209 ff.) does a similar analysis.

which does not change. While this case is very simple, it is not totally unrealistic. It is often used in empirical analysis, especially when a separability assumption between the good whose price changes and the other goods is appropriate. A very general treatment of separability is contained in Blackorby, Primont, and Russell, but for use herein, a simple interpretation of separability which allows us to write the utility function of equation (1) as $u(x_1,\ldots,x_n)=u(x_1,g(x_2,\ldots,x_n))$ is adequate. The appropriate price index which corresponds to the structure of $u(\cdot)$ provides the numeraire good. Separability of the indirect utility function is defined in an analogous manner, $v(p_1,k(p_2,\ldots,p_n)$ where $k(\cdot)$ provides the price index. In general separability of $u(\cdot)$ does not imply separability of $v(\cdot)$ or vice versa.

Separability utility functions justify specification and estimation of demand curves that have only a single price in them. An important example often used in empirical studies is the linear labor supply relationship

$$(11) \quad x_j=\alpha w_j+\delta y_j+z_j\gamma+\varepsilon_j; \quad j=1,\ldots,J$$

estimated over a sample of J individuals where w_j is the commodity price deflated (net after tax) wage, y_j is the commodity price deflated nonlabor income, Z_j is a vector of socioeconomic characteristics, and ε_j is a stochastic disturbance. Numerous other commodity demand equations are specified in this form where the wage is replaced by the price of the commodity.

To derive the exact compensating variation is straightforward and provides an exact welfare measure. The basic idea is to take the *observed* market demand curve and to use Roy's identity from equation (5) to integrate and derive the indirect utility function.[10] Inversion of the indirect utility gives the expenditure function which allows calculation of the compensating variation. Equivalently, using equation (3) we can derive the *unobservable* compensated demand curve. And equation (8) shows that the area under the compensated demand curve yields the exact consumer's surplus.

In principle we can always perform this integration for a well-specified demand function. This statement is the essence of the famous integrability problem in consumer demand.[11] So long as the derivatives of the *compensated* demand functions satisfy the properties of symmetry and negative semidefiniteness of the Slutsky matrix and the adding-up condition, the indirect utility function can be recovered by integration.[12] In practice, many commonly used demand functions in empirical work yield explicit solutions so that exact welfare analysis is easily done.

Returning to the two-good example, consider the nonstochastic demand function (where both p_1 and y are deflated by the price of the other good, p_2):[13]

$$(12) \quad x_1=\alpha p_1+\delta y+z\gamma$$
$$=-\partial v(p_1,y)/\partial p_1/\partial v(p_1,y)/\partial y$$

I solve this linear partial differential equation by applying the method of characteristic curves which assures a unique solution, given an initial condition.[14] To make welfare comparisons we will want to be on a given indifference curve. As the price changes I will use the equation $v(p_1(t),y(t))=u_0$ for some u_0; for example, initial utility in the compensat-

[10] This technique has been used in estimating demand with nonlinear budget constraints by Gary Burtless and myself, and in my earlier article.

[11] See Samuelson and Hurwicz and Uzawa.

[12] In addition a regularity condition is needed. A Lipschitz-type condition is given by Hurwicz and Uzawa. A stronger sufficient condition that often holds is for the demand function to be continuously differentiable.

[13] It has been pointed out to me by Diewert that this demand specification corresponds to a flexible functional form for the underlying preferences as discussed in Blackorby and Diewert. Basically, three independent parameters are needed for the demand function in the two-good case, which equation (12) has, so that the value of demand, the uncompensated price derivative, and the income derivative can attain arbitrary values.

[14] See Fritz John or Richard Courant and David Hilbert. Given that along an initial curve (here an indifference curve), the initial values are continuously differentiable then a unique solution to the partial differential equation exists.

ing variation case. Along a path of price change to stay on the indifference curve, we have

$$(13)\qquad \frac{\partial v(p_1(t), y(t))}{\partial p_1(t)}\frac{dp_1(t)}{dt} + \frac{\partial v(p_1(t), y(t))}{\partial y(t)}\frac{dy(t)}{dt} = 0$$

Then, using the implicit function theorem and Roy's identity from equation (12),

$$(14)\qquad \frac{dy(p_1)}{dp_1} = \alpha p_1 + \delta y + z\gamma$$

I have now expressed y as a function of p_1 and can solve the ordinary differential equation (14) to find

$$(15)\quad y(p_1) = c e^{\delta p_1} - \frac{1}{\delta}\left(\alpha p_1 + \frac{\alpha}{\delta} + z\gamma\right)$$

where c, the constant of integration, depends on the initial utility level u_0. In fact, I simply choose $c = u_0$ as our cardinal utility index. Therefore, solving equation (15), we find the indirect utility function[15]

$$(16)\qquad v(p_1, y) = c = e^{-\delta p_1}\left[y + \frac{1}{\delta}\left(\alpha p_1 + \frac{\alpha}{\delta} + z\gamma\right)\right]$$

Then the corresponding expenditure function (again normalized by the price of the second good) follows simply from equation (16) by interchanging the utility level with the income variable

$$(17)\quad e(p_1, \bar{u}) = e^{\delta p_1}\bar{u} - \frac{1}{\delta}\left(\alpha p_1 + \frac{\alpha}{\delta} + z\gamma\right)$$

It is important to note that this procedure yields a *local solution* to the differential equation over some domain in price space. It is not always the case that there exists a global solution to equation (12) which satisfies the integrability conditions. However, we need only a local solution to make the welfare calculations that we are interested in. That is, we only want to compute a welfare measure at two price points, sat p_1^0 and p_1^1, which equations (16) and (17) permit us to do.

We now have a solution to Roy's identity, but we need to check whether we have a valid indirect utility function which arises from consumer maximization.[16] The indirect utility function of equation (16) is continuous and homogeneous of degree zero in prices and income by my normalization condition using p_2 as numeraire. It is also decreasing in prices if $\alpha \leq 0$ and increasing in income if $\delta \geq 0$. The other condition $v(p_1, y)$ must satisfy is quasi concavity which is equivalent to the Slutsky condition

$$(18)\qquad s_{11} = \frac{\partial h_1(p_1, \bar{u})}{\partial p_1} = \alpha + \delta(\alpha p_1 + \delta y + z\gamma) \leq 0$$

where the compensated demand curve $h_1(p_1, \bar{u})$ follows from the expenditure function of equation (17) by differentiation with respect to p_1. So long as the sign conditions are satisfied by the demand function we can calculate exact consumer's surplus and deadweight loss using the expenditure function of equation (17) and indirect utility function of equation (16).

To compute the compensating variation we use equation (17) and equation (6) to find

$$(19)\quad CV(p_1^0, p_1^1, y_0) = e^{\delta(p_1^1 - p_1^0)}\left[y_0 + \frac{1}{\delta}\left(z\gamma + \frac{\alpha}{\delta} + \alpha p_1^0\right)\right] - \frac{1}{\delta}\left(z\gamma + \frac{\alpha}{\delta} + \alpha p_1^1\right) - y^0$$
$$= \frac{1}{\delta}e^{\delta(p_1^1 - p_1^0)}\left[x_1^0(p_1^0 \cdot y_0) + \frac{\alpha}{\delta}\right] - \frac{1}{\delta}\left[x_1^1(p_1^1 \cdot y^0) + \frac{\alpha}{\delta}\right]$$

[15]Any monotonic transformation of this equation will of course satisfy the differential equation since ordinal utility is determined only up to a monotonic transformation. The only change would be in c, the constant of integration.

[16]Diewert discusses the appropriate conditions.

This expression for the compensating variation, while certainly more complicated than the Marshallian triangle formula, is still straightforward to calculate. The corresponding equivalent variation would be calculated from equation (7). Furthermore, since the parameters for equation (17) are presumably estimated by econometric methods, well-known methods allow calculation of the large sample standard error for the compensating variation in equation (19) (for example, see Rao, p. 323). Note, also that the compensating variation now varies across individuals by their socioeconomic characteristics and their income levels while the corresponding Marshallian expressions neglects these factors in its approximation. Use of the compensating variation or equivalent variation ends all arguments about the appropriateness of the Marshallian approximation since they give the exact measure of welfare change.

Another commonly used demand curve specification in the two-good case is the constant elasticity specification[17]

$$(20)\quad x_1 = e^{z\gamma} p_1^{\alpha} y^{\delta} = -\partial v(p_1, y)/\partial p_1 / \partial v(p_1, y)/\partial y \qquad \delta \neq 1$$

which is often estimated in *log*-linear form as $\log x_{1j} = z_j\gamma + \alpha \log p_{1j} + \delta \log y_j + \varepsilon_j$ for $j = 1, \ldots, J$.[18] To find the indirect utility function we use the technique of separation of variables and integrate to find

$$(21)\quad v(p_1, y) = c = -e^{z\gamma} \cdot \frac{p_1^{1+\alpha}}{1+\alpha} + \frac{y^{1-\delta}}{1-\delta}$$

where c, the constant of integration, has again been set at the initial utility level. The Slutsky condition is $s_{11} = x_1(\alpha/p_1 + \delta x_1/y)$. The expenditure function (again normalized by p_2) is

$$(22)\quad e(p_1, \bar{u}) = \left[(1-\delta)\left(\bar{u} + e^{z\gamma}\frac{p_1^{1+\alpha}}{1+\alpha}\right)\right]^{1/1-\delta}$$

so that the compensating variation for a change in price from p_1^0 to p_1^1 is the quantity

$$(23)\quad CV(p_1^0, p_1^1, y^0) = \left\{(1-\delta)\left[\frac{e^{z\gamma}}{1+\alpha}\left(p_1^{1\,1+\alpha} - p_1^{0\,1+\alpha}\right)\right] + y^{0(1-\delta)}\right\}^{1/1-\delta} - y^0$$

$$= \left\{\frac{(1-\delta)}{(1+\alpha)y^{0\delta}}\left[p_1^1 x_1^1(p_1^1, y^0) - p_1^0 x_1^0(p_1^0, y^0)\right] + y^{0(1-\delta)}\right\}^{1/1-\delta} - y^0$$

Again an exact formula for the compensating variation is derived for which a standard error could be straightforwardly calculated given a covariance matrix for the estimated parameters. No approximation argument is required in using the compensating variation as a measure of welfare change. It is interesting to note that while the denominator of equation (9) is constant for the demand specification of equation (20) so that in this case the Marshallian area also gives an exact measure of welfare change, it is not equal to either the compensating variation or the equivalent variation. The income effect from equation (10) is not zero so that the compensated demand derivative and uncompensated demand derivative differ by a positive amount. Thus, use of the Marshallian measure still involves an error of approximation if either the compensating variation or the equivalent variation are the desired measure.

II. The Many-Good Case and More General Demand Specifications

The welfare measures developed at the beginning of Section I were all fully general in the sense that they considered n different

[17]Again this demand curve provides a flexible functional form for the underlying preferences.

[18]Willig considers a constant income elasticity demand specification in deriving his approximations. For $\delta = 1$ the indirect utility function has the same form as equation (19) except that the last term is replaced by $\log y$.

goods and allowed all n prices to change. In particular, the compensating variation of equation (6) and the equivalent variation of equation (7) used the expenditure function whose arguments are the complete price vector and the appropriate utility level. In this section I generalize the methods of calculating the compensating variation to the many-good case but continue to consider only one price change.[19] While we cannot recover the complete expenditure function as before, we can still recover the quasi-expenditure function whose derivative yields the appropriate compensated demand curve. Thus again the compensating variation and equivalent variation can be estimated exactly given information on the market demand curve for the good whose price has changed.

A complete specification of a system of demand equations would have the general form

$$(24)\quad x_i = x(p, y, z, \varepsilon_i); \qquad i = 1, \ldots, N$$

where p is the price vector, z is a vector of socioeconomic characteristics, and ε_i is a stochastic disturbance. So long as the estimated coefficients of the demand system have the property that the Slutsky matrix is symmetric and negative semidefinite and that the function $x(\cdot)$ is regular in p and y, then in principle the system can be integrated and the expenditure functions derived. However, the usual case is that we do not have information on all quantity demands at the individual level. But suppose we do have information on demand for, say, the first good whose price is expected to change as a result of the public policy measure being considered. A first-order Taylor expansion of equation (24) would lead to the econometric specification[20]

$$(25)\quad x_1(p, y) = z\gamma + \sum_{i=2}^{N} \frac{\delta_i y}{p_i} + \sum_{i=2}^{N} \frac{\alpha_i p_1}{p_i} + \varepsilon_1$$

The important point to note about equation (25) is that by assumption only p_1 will change due to the contemplated policy measure, while z, y, and $p_2, \ldots, p_n$ will remain constant. Thus, all prices except the first can be written as a scalar multiple of a price index, $p_2 = \lambda_2 q, \ldots, p_N = \lambda_N q$ where $\lambda_2, \ldots, \lambda_N$ are known fixed positive constants. We can now apply Hicks' composite commodity theorem.[21] Rewrite equation (25) as

$$(26)\quad x_1(p_1, q, y) = z\gamma + \left(\sum_{i=2}^{N} \frac{\delta_i}{\lambda_i}\right)\frac{y}{q} + \left(\sum_{i=2}^{N} \frac{\alpha_i}{\lambda_i}\right)\frac{p_1}{q} = z\gamma + \delta\frac{y}{q} + \alpha\frac{p_1}{q}$$

where $\delta = \sum_{i=2}^{N} \delta_i/\lambda_i$ and $\alpha = \sum_{i=2}^{N} \alpha_i/\lambda_i$

Since equation (26) is the same as equation (12) except that the composite price q has replaced p_2, I can repeat the analysis of the last section with the welfare analysis based on equations (16) and (17). Note that the resulting functions might best be referred to as a quasi-indirect utility function and a quasi-expenditure function. We have not recovered the complete indirect utility function or expenditure function, but the "quasi" functions lead to exact welfare measures when all other prices are constant. But they cannot be used to analyze the welfare change when more than one price changes (except proportionately) without further analysis.

Let us now briefly consider some extensions of our techniques to more general cases. First, we can generalize the *log*-linear demand specification of equation (20) to the many good consumer

$$(27)\quad x_1(p, y) = e^{z\gamma} \prod_{i=2}^{N} \left(\frac{p_1}{p_i}\right)^{\alpha_i} \prod_{i=2}^{N} \left(\frac{y}{p_i}\right)^{\delta_i}$$

[19] The one-price-change situation is the case considered by Willig.

[20] I am indebted to Diewert for help in improving this section of the paper from an earlier version.

[21] For other references and developments of this theorem, see Terrance Gorman and Blackorby et al. and Diewert.

Again, if only the first price changes, we can obtain the quasi-expenditure function corresponding to equation (22) by the application of Hicks' composite commodity theorem to obtain

$$(28)\quad x_1(p_1,q,y)=e^{z\gamma}\left(\frac{p_1}{q}\right)^{\sum_{i=2}^{N} a_i}\left(\frac{y}{q}\right)^{\sum_{i=2}^{N} \delta_i}$$

Use of the quasi-expenditure function allows exact welfare measures to be calculated.

I now return to the two-good case to present some generalizations of the demand specification with the observation that they can be expanded to the N good case by the techniques which lead to equations (26) and (28). Thus, I again normalize by the second price so that p_1 and y are divided by p_2. I return to the linear demand specification of equation (12) but allow the price and income coefficients of the demand specification, as well as the intercept, to depend on individual socioeconomic characteristics. Let $\delta=zd$ and $\alpha=za$ which leads to the demand specification[22]

$$(29)\quad x_1(p_1,y)=z\gamma+zdy+zap_1+\varepsilon_1$$

Calculation of the welfare measures proceeds in the same way except that δ and α vary across individuals. Perhaps a more important generalization is to allow interactions among the price terms to move away from the linear demand curve specification. A demand function quadratic in prices is

$$(30)\quad x_1(p_1,y)=z\gamma+\delta y+\beta_1 p_1+\beta_2 p_1^2+\varepsilon_1$$

so long as the Slutsky term is negative we can integrate the corresponding differential equation by parts to find the indirect utility function

$$(31)\quad v(p_1,y)=e^{-\delta p_1}\left[y+a_1 p_1+a_2 p_1^2+a_3\right]$$

where $a_1=\beta_1/\delta+2\beta_2/\delta^2$, $a_2=\beta_2/\delta$, and $a_3=z\gamma/\delta+2\beta_2/\delta^3$. With equation (31) exact welfare analysis is again straightforward since the expenditure function, compensating variation, and equivalent variation all follow from equation (31).

The last and most general demand curve that is considered is fully quadratic in both prices and income. The demand function is

$$(32)\quad x_1=\beta_0+\beta_1 y+\beta_2 p_1+\beta_3 y^2+\beta_4 p_1^2+\beta_5 p_1 y+\varepsilon_1$$

where $\beta_0=z\gamma$. Using Roy's identity we have the nonlinear differential equation

$$(33)\quad \frac{dy}{dp_1}+Qy+Ry^2+S=0$$

where $R=-\beta_3$, $Q=-(\beta_1+\beta_5 p_1)$ and $S=-(\beta_0+\beta_2 p_1+\beta_4 p_1^2)$. It turns out that this equation can be transformed by changes of variables to the famous Schrodinger wave equation of physics. I give the derivation in the Appendix where the indirect utility function is found to have the form

$$(34)\quad v(p,y)=\left(h\tilde{W}_1-\tilde{W}_1^1\right)/\left(\tilde{W}_2^1-h\tilde{W}_2\right)$$

where $h=-\beta_3 y+(\beta_5/2)(\beta_1+\beta_5 p)^2$ and $\tilde{W}_1$ and $\tilde{W}_2$, functions of the β parameters of equation (32) and prices, which are straightforward to calculate. Their exact form is given in the Appendix. Again, the expenditure function and exact welfare measures follow directly from equation (34). Thus, we have a very general demand specification with associated exact welfare measures. In fact, the demand function may well provide a third-order flexible function form in the sense of Blackorby and Diewert.

III. Calculation of the Compensating Variation and of the Deadweight Loss

In the previous section, I have given formulae for calculating the exact welfare change by deriving the unobservable compensated demand curve given market information. Here I consider two examples to demonstrate use of the formulae. I can also assess how accurate the Marshallian ap-

[22]Stochastic terms can be added of the type $\delta=Zd+v$, which lead to a random coefficients specification. The resulting heteroscedasticity can be accounted for in the estimation procedure. This type of demand function is estimated in my article with Burtless.

proximations are for the exact welfare measures. The first example of labor supply shows that the approximation may be quite poor for goods which form a large proportion of total expenditure. Since Willig showed that the approximation might not do well in this case, the finding is not surprising. However, the second example raises severe doubt about the use of uncompensated market demand for a commodity which is only a small proportion of the budget when we calculate the deadweight loss from the imposition of a tax. Even though the conditions for an accurate approximation to the compensating variation hold, the approximation to the deadweight loss is very inaccurate. In fact, this finding seems to hold in general. While the Marshallian approximation is adequate in certain situations for the compensating variation, it is often *not* accurate under these conditions for measurement of the deadweight loss. Since measurement of the deadweight loss is often the goal in applied welfare economics, this finding strongly recommends use of the exact measure deadweight rather than the Marshallian approximation.

The first example used, is a linear labor supply function of the form of equation (11). The estimates used are taken from a study of wives' labor supply functions in my forthcoming paper. The estimated values used for the jth individual are

$$(35)\quad h_j = 495.1 w_j - .1250 y_j + 765.1$$

The left-hand side variable is hours per year of work, w_j is market wage which has a mean of \$4.15 per hour, y_j is after tax income of the husband which has a mean of \$8,236, and the constant takes account of demographic factors such as age and children.

Here I calculate the required compensating variation after the imposition of a 20 percent proportional tax on labor earnings. Compared to a no-tax situation, the expenditure takes the form

$$(36)\quad e(w,\bar{u}) = e^{-\delta w}\bar{u} - \frac{\alpha}{\delta}w + \frac{\alpha}{\delta^2} - \frac{z\gamma}{\delta}$$

Calculating u^0 from the corresponding indirect utility function and using it in equation (36) leads to a required expenditure of \$9485 per year. I find that the compensating variation is \$2,056. Using the formula for distribution of a nonlinear function, I find one standard error for the compensating variation to be plus or minus \$481. Then to find the aggregate compensating variation for the complete population, a sample enumeration would be done allowing the wages, husband's income, and socioeconomic variables to differ across individuals.

Calculations of the Marshallian approximation is straightforward since we use the estimates of equation (35) and measure the area to the left of the labor supply curve between the initial and final net wages of \$4.15 per hour and \$3.32 per hour. The Marshallian approximation is \$1,315 per year so that the two measures differ by 44.6 percent. Thus, the Marshallian measure provides a very poor approximation to the exact measure of welfare change. That the Marshallian measure provides a poor approximation in this case is in line with Willig's results since the Marshallian area is large with respect to base income. Hence, the Taylor approximation which provides Willig's bounds demonstrates that the derivation between the two measures can be substantial. It is worth emphasizing again that the exact welfare change is easily calculated from the indirect utility function and the expenditure function. Then no worry about the accuracy of the approximation is needed.

The last example I consider is the more important one, since it involves a quite common use of consumer's surplus in applied welfare economics. I consider the deadweight loss from imposition of a commodity tax.

Consider the compensated demand curve $h(p,u_0)$ shown in Figure 2. The compensating variation is the area to the left of the demand curve between the initial price p^0 and the final, post tax, price p^1. But we are often *more* interested in the welfare triangle which measures the efficiency loss from the use of distorting taxes. This triangle corresponds to the Harberger measure. Therefore, I define the deadweight loss to be the difference of the compensating variation minus

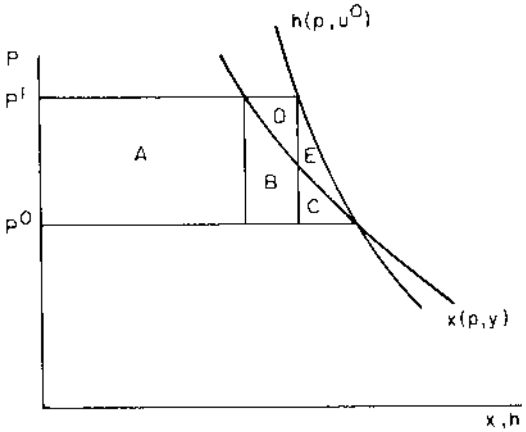

FIGURE 2

the tax revenue collected. The rectangle in Figure 2 thus has only distributional consequences while the triangle is the deadweight burden which cannot be undone. Optimal tax policy typically tries to minimize the sum of the deadweight losses to achieve a second best optimum, for example, see Diamond and Mirrlees.

The particular example I consider is meant to approximate the long-run demand for gasoline, although the numbers used are hypothetical. The demand function is

$$(37) \qquad q_j = -14.22p + .082y_j + 4.95$$

Choosing income for the mean person to be \$720 per month and initial price to be \$.75 per gallon, the price elasticity is .2 with an income elasticity of 1.1. Both elasticities are similar to elasticities which have been found in empirical studies. Let us now consider imposition of a tax which raises the price of gasoline to \$1.50 per gallon. Using equation (17) we find that the compensating variation equals \$37.17 per month. The Marshallian approximation equals \$35.99 per month, so that the two measures differ by only 3.2 percent. Thus, the Willig results are confirmed since demand for gasoline is only a small part of the total budget for the individual.

However, when we compare the two measures of deadweight loss we find a substantial difference. The compensated measure of the deadweight loss is \$2.88 while the Marshallian measure is \$3.96. The two measures differ by 31.7 percent, even though the approximation is good for the compensating variation. Why can the approximation be so poor for the deadweight loss? Using order arguments somewhat loosely, the compensating variation is composed of two pieces, the rectangle which is a first-order quantity of demand times change in price while the deadweight loss is a second-order quantity of one-half the changes in demand times the change in price. While the Marshallian approximation does reasonably well for the first-order part of the compensating variation under certain conditions given by Willig, its performance on the second order part may still be quite bad.

In Figure 2 we see that both measures of the compensating variation have rectangle A in common, which is a large part of the whole. In measuring the first-order effect they differ only by triangle D, which is small compared to the whole. However, in measuring the deadweight loss, the percentage difference will depend on the difference of area B and triangle E compared to the area of triangle C. Figure 2 shows that this difference can often be substantial. Thus, the Marshallian approximation is not accurate for measurement of the deadweight loss. Instead, the exact Hicksian measure should be used. While the Willig results will hold for the compensating variation, if the goal of the calculation is deadweight loss, the Marshallian approximation should not be used. In many cases it is a very inaccurate measure of the true deadweight loss.

IV. Conclusion

In empirical situations where a measure of either the compensating variation, equivalent variation, or deadweight loss is needed, economists often work with relatively simple demand specifications. For these types of specifications we have developed the exact measures of welfare change. While it has been known that use of the compensated demand curves lead to the appropriate welfare measures, it has not been generally recognized how straightforward it is to derive the compensated demand curves from observed market demand curves. I derived methods which are easily applied to the two-good case. These methods are then extended to the many-good case with one price change. The quasi-indirect utility function and expenditure function provide the appropriate compensated demand curve and thus the appropriate welfare measure. While our measures tell us the appropriate compensation, they, of course, do not necessarily give the correct measurement of the loss in social welfare if no compensation is paid.

Through two examples I attempt to assess the accuracy of the Marshallian approximation. For a good which forms a small part of the total budget, the Marshallian area is reasonably accurate as proven by Willig. But if the good forms a large part of the budget, the approximation may be quite inaccurate as our labor supply example shows. A more important finding is the high level of inaccuracy when the deadweight loss, or welfare triangle, is measured. For deadweight loss, the Marshallian area can often be quite far off even though it is reasonably accurate for the compensating variation in the same situation. The gasoline example shows that the deadweight loss measures differ by 32 percent even though the compensating variation measures differ by 3.2 percent. Thus, it seems *inappropriate* to measure deadweight loss by using the market demand curve. But since the exact deadweight loss measure can be often calculated by use of the compensated demand curve, no special problem arises. The formulae given in this paper permit exact calculation of both the compensating variation and of the deadweight loss.

APPENDIX

Let us consider derivation of the indirect utility function and expenditure function which corresponds to the fully quadratic demand curve of Section II.[23]

[23]Generalization to the many-good case is straightforward. Only a sketch of the derivation is provided here. Further details may be obtained by writing the author.

The demand function that I consider is

$$x_1=\beta_0+\beta_1 y+\beta_2 p_1+\beta_3 y^2+\beta_4 p_1^2 +\beta_5 p_1 y+\varepsilon_1$$

where $\beta_0=Z\gamma$. Using Roy's identity this demand equation may be written as the nonlinear differential equation

$$y'+Qy+Ry^2+S=0$$

where $R=-\beta_3$, $Q=-(\beta_1+\beta_5 p_1)$ and $S=-(\beta_0+\beta_2 p_1+\beta_4 p_1^2)$. I do one change of dependent variable $y=(1/R)(u'/u)$ and one change of independent variable $t=\beta_1+\beta_5 p_1$ calling the resulting function $\phi(t)$ to find

$$\phi''+\beta_5 t\phi'+q\phi=0$$

where $q=\beta_5^2 SR$. Thus, I have transformed the nonlinear equation, a Ricatti equation, to a second-order differential equation of the form studied by physicists. I then transform by $W=\phi e^{\beta_5 t^2/4}$ to put the equation in parabolic cylinder form $W''+WM=0$ where $M=\delta_0+\delta_1 t+\delta_2 t^2$ and the δ_i's are easily calculated functions of the β_i's. I have thus transformed the original equation into the famous Schrodinger wave equation. One last change of independent variable $x^2=4(\delta_1 t+\delta_2 t^2)$ and we have the final form

$$W''+W\big(\delta_0+(x^2/4)\big)=0.$$

Define the functions $W_1=1+\delta_0(x^2/2)+(\delta_0^2-(1/2)(x^4/4!)+\ldots$ and $W_2=x+\delta_0(x^3/3!)+(\delta_0^2-3/2)(x^5/5!)+\ldots$, which converge quickly for values likely to be encountered in economics.[24] Now define $\gamma_0=\delta_1\beta_1+\delta_2\beta_1^2$, $\gamma_2=\delta_1\beta_5+2\delta_2\beta_1\beta_5$, and $\gamma_3=\delta_2\beta_5^2$ and we have the $\tilde{W}_i$ function in terms of prices $\tilde{W}_1=1+2\delta_0(\gamma_1+\gamma_2 p_1+\gamma_3 p_1^2)+(2/3)(\delta_0^2-1/2)(\gamma_1+\gamma_2 p_1+\gamma_3 p_1^2)^2+\ldots$ and $\tilde{W}_2=2\delta_0(\gamma_2+2\gamma_3 p_1)+(1/3)(\delta_0^2-1/2)(\gamma_1+\gamma_2 p_1+\gamma_3 p_1^2)(\gamma_2+2\gamma_3 p_1)+\ldots$ which again converge

[24]Description and analysis of the parabolic cylinder functions is found in Milton Abramowitz and Irene Stegum (ch. 19). The successive coefficients of the expansion have a simple recursive formula which eases calculation.

quickly. The indirect utility function thus takes the form

$$v(p,y)=\frac{h\tilde{W}_1-\tilde{W}_1'}{\tilde{W}_2'-h\tilde{W}_2} \tag{A1}$$

where $h=-\beta_3 y+\beta_5 t^2/2$. The expenditure function also takes a simple form in terms of the W functions

$$e(p_1,\bar{u})=\frac{t\beta_5^2}{2\beta_3}-\frac{1}{\beta_3}\left(\frac{\tilde{W}_1'+\bar{u}\tilde{W}_2'}{\tilde{W}_1+\bar{u}\tilde{W}_2}\right) \tag{A2}$$

Then equation (A1) is used to compute utility at original prices p_1^0 and equation (A2) is used to compute $e(p_1^1,\bar{u})$ so that after subtracting off y_0 we find the compensating variation. The W functions are straightforward to calculate and both tables and computer routines exist to do the calculation.

I might note that it is straightforward to generate demand functions and corresponding indirect utility functions and expenditure functions that are closed form and contain quadratic terms in both prices and incomes. But I have not yet found demand functions of this type which can be estimated using linear regression techniques. The specification leading to (A1) and (A2) has this advantage although specialized computer routines then become necessary to evaluate the consumer's surplus and deadweight loss measures.

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